HYPERSURFACES WITH NULL HIGHER ORDER ANISOTROPIC MEAN CURVATURE

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ABSTRACT. Given a positive function F on $\mathbb{S}^n$ which satisfies a convexity condition, for $1 \le r \le n$, we define for hypersurfaces in $\mathbb{R}^{n+1}$ the r-th anisotropic mean curvature function $H_{r;F}$, a generalization of the usual r-th mean curvature function. We call a hypersurface is anisotropic minimal if $H_F = H_{1;F} = 0$, and anisotropic r-minimal if $H_{r+1;F} = 0$. Let W be the set of points which are omitted by the hyperplanes tangent to M. We will prove that if an oriented hypersurface M is anisotropic minimal, and the set W is open and non-empty, then $x(M)$ is a part of a hyperplane of $\mathbb{R}^{n+1}$. We also prove that if an oriented hypersurface M is anisotropic r-minimal and its r-th anisotropic mean curvature $H_{r;F}$ is nonzero everywhere, and the set W is open and non-empty, then M has anisotropic relative nullity $n - r$.

1. Introduction

Let $F : \mathbb{S}^n \to \mathbb{R}^+$ be a smooth function which satisfies the following convexity condition:

$$(D^2F + FI)_x > 0, \quad \forall x \in \mathbb{S}^n, \tag{1.1}$$

where $\mathbb{S}^n$ is the standard unit sphere in $\mathbb{R}^{n+1}$, D^2F denotes the intrinsic Hessian of F on $\mathbb{S}^n$ and I denotes the identity on $T_x\mathbb{S}^n$, > 0 means that the matrix is positive definite. We consider the map

$$\begin{aligned} &\phi : \mathbb{S}^n \to \mathbb{R}^{n+1}, \\ &x \to F(x)x + (\operatorname{grad}_{\mathbb{S}^n} F)_x, \end{aligned} \tag{1.2}$$

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its image $W_F = \phi(\mathbb{S}^n)$ is a smooth, convex hypersurface in $\mathbb{R}^{n+1}$ called the Wulff shape of F (see [6], [8], [17], [18], [19], [20], [22], [24], [29]). When $F \equiv 1$, the Wulff shape W_F is just $\mathbb{S}^n$.

Now let $x : M \to \mathbb{R}^{n+1}$ be a smooth immersion of an oriented hypersurface. Let $N : M \to \mathbb{S}^n$ denote its Gauss map. The map $\nu = \phi \circ N : M \to W_F$ is called the anisotropic Gauss map of x.

Let $S_F = -\mathrm{d}\nu$. S_F is called the F-Weingarten operator, and the eigenvalues of S_F are called anisotropic principal curvatures. Let σ_r be the elementary symmetric functions of the anisotropic principal curvatures $\kappa_1, \kappa_2, \cdots, \kappa_n$:

$$\sigma_r = \sum_{i_1<\cdots<i_r} \kappa_{i_1} \cdots \kappa_{i_r} \quad (1 \le r \le n).$$

We set $\sigma_0 = 1$. The r-th anisotropic mean curvature $H_{r;F}$ is defined by $H_{r;F} = \sigma_r / C_n^r$, also see Reilly [25]. $H_F := H_{1;F}$ is called the anisotropic mean curvature. When $F \equiv 1$, S_F is just the Weingarten operator of hypersurfaces, and $H_{r;F}$ is just the r-th mean curvature H_r of hypersurfaces which has been studied by many authors (see [9], [21], [23], [27]). Thus, the r-th anisotropic mean curvature $H_{r;F}$ generalizes the r-th mean curvature H_r of hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$.

We say that $x : M \to \mathbb{R}^{n+1}$ is anisotropic r-minimal if $H_{r+1;F} = 0$.

For $p \in M$, we define $v(p) = \dim\ker(S_F)$. We call $v = \min_{p\in M} v(p)$ the anisotropic relative nullity, it generalized the usual relative nullity.

For a smooth immersion $x : M \to \mathbb{Q}_c^{n+1}$ of a hypersurface into an $(n+1)$-dimensional space form with constant sectional curvature c, we denote by

$$W = \mathbb{Q}_c^{n+1} - \bigcup_{p\in M} (\mathbb{Q}_c^n)_p,$$

where for every $p \in M$, $(\mathbb{Q}_c^n)_p$ is the totally geodesic hypersurface of $\mathbb{Q}_c^{n+1}$ tangent to $x(M)$ at $x(p)$. So, in the case of $c = 0$, W is the set of points which are omitted by the hyperplanes tangent to $x(M)$.

We will study immersion whose set W is nonempty. In this direction, T. Hasanis and D. Koutroufiots, see [14], proved that

Theorem 1.1. *Let $x : M \to \mathbb{Q}_c^3$ be a complete minimal immersion with $c \ge 0$. If W is nonempty, then x is totally geodesic.*

Later, in [3], H. Alencar and K. Frensel extended the result above assuming an extra condition. They proved that

Theorem 1.2. *Let $x : M \to \mathbb{Q}_c^{n+1}$ be an oriented, minimally immersed hypersurface. If W is open and non-empty then x is totally geodesic.*

In [2], H. Alencar and M. Batista studied hypersurfaces with null higher order mean curvature, they proved

Theorem 1.3. *Let M be a complete and orientable Riemannian manifold and let $x : M \to \mathbb{Q}_c^{n+1}$ be an isometric immersion with $H_{r+1} = 0$ and $H_r \neq 0$ everywhere, $r \geq 1$. If W is open and nonempty, then the relative nullity $v = n - r$.*

We note that, H. Alencar in [1] provides examples of non-totally geodesic minimal hypersurfaces in $\mathbb{R}^{2n}$, $n \geq 4$, with nonempty W; in [2], H. Alencar and M. Batista provides examples of 1-minimal hypersurfaces with $H_1 \neq 0$ everywhere in $\mathbb{R}^{2n}$, $n \geq 5$, with nonempty W but $v \neq n - 1$. These examples show that is necessary to add an extra hypothesis.

In this paper, we prove the anisotropic version of Theorem 1.2 and Theorem 1.3 for an immersion $x : M \to \mathbb{R}^{n+1}$, we prove:

Theorem 1.4. *Let $x : M \to \mathbb{R}^{n+1}$ be an oriented, anisotropic minimally immersed hypersurface. If W is open and non-empty then $x(M)$ is a part of a hyperplane of $\mathbb{R}^{n+1}$.*

Theorem 1.5. *Let $x : M \to \mathbb{R}^{n+1}$ be an oriented immersed hypersurface with $H_{r+1;F} = 0$ and $H_{r;F} \neq 0$ everywhere, $r \geq 1$. If W is open and nonempty, then the anisotropic relative nullity $v = n - r$.*

2. Preliminaries

In this paper, we use the summation convention of Einstein and the following convention of index ranges unless otherwise stated:

$$1 \leq i, j, \cdots \leq n; \quad 1 \leq \alpha, \beta, \cdots \leq n + 1.$$

We define $F^* : \mathbb{R}^{n+1} \to \mathbb{R}$ to be:

$$F^*(y) = \sup \left\{ \frac{\langle y, z\rangle}{F(z)} | \ z \in \mathbb{R}^{n+1} \setminus \{0\} \right\}, \tag{2.1}$$

then F^* is a Minkowski norm on $\mathbb{R}^{n+1}$. In fact, as proved in [15], $F^* : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is smooth and we have

Proposition 2.1. (1) *$F^*(y) > 0$, $\forall y \in \mathbb{R}^{n+1} \setminus \{0\}$;*
(2) *$F^*(ty) = tF^*(y)$, $\forall y \in \mathbb{R}^{n+1}, t > 0$;*

(3) $F^*(y+z) \le F^*(y) + F^*(z)$, $\forall y, z \in \mathbb{R}^{n+1}$, *and the equality holds if and only if* $y = 0$, *or* $z = 0$ *or* $y = kz$ *for some* $k > 0$.
(4) $W_F = \{y \in \mathbb{R}^{n+1} | \ F^*(y) = 1\}$.

We define

$$\bar{g}_{\alpha\beta}(y) = \frac{1}{2}\frac{\partial^2 (F^*)^2}{\partial y^\alpha \partial y^\beta}(y), \tag{2.2}$$

and

$$g_y(X,Y) = \bar{g}_{\alpha\beta}(y) X^\alpha Y^\beta, \tag{2.3}$$

where $y \in \mathbb{R}^{n+1} \setminus \{0\}$ and $X = (X^1, X^2, \cdots, X^{n+1})$, $Y = (Y^1, Y^2, \cdots, Y^{n+1}) \in T_y\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$.

From the Euler's theorem for homogeneous functions, we have

$$\frac{\partial \bar{g}_{\alpha\beta}}{\partial y^\gamma}(z) z^\beta = \frac{1}{2}\frac{\partial^3 (F^*)^2}{\partial y^\alpha \partial y^\beta \partial y^\gamma}(z) z^\beta = 0,$$

where $z = (z^1, z^2, \cdots, z^{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$. Thus,

$$\frac{\partial g_z(X,z)}{\partial y^\gamma} = \bar{g}_{\alpha\beta}(z)\frac{\partial X^\alpha}{\partial y^\gamma} z^\beta + \bar{g}_{\alpha\gamma}(z) X^\alpha \frac{\partial z^\beta}{\partial y^\gamma}, \tag{2.4}$$

where $z = (z^1, z^2, \cdots, z^{n+1}) \in T\mathbb{R}^{n+1}$ is nonzero everywhere and $X = (X^1, X^2, \cdots, X^{n+1}) \in T\mathbb{R}^{n+1}$.

As F^* is a Minkowski norm on $\mathbb{R}^{n+1}$, the following lemma holds (see [4], [28]):

Lemma 2.2. *For any* $y \in \mathbb{R}^{n+1} \setminus \{0\}$ *and* $u \in \mathbb{R}^{n+1}$ *we have*

$$g_y(y,z) \le F^*(y) F^*(z), \tag{2.5}$$

and equality if and only if there exists $t \ge 0$ *such that* $z = ty$.

Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Let $\nu : M \to W_F$ denote its anisotropic Gauss map. Then for any $p \in M$, $\nu(p)$ is perpendicular to $x_*(T_pM)$ with respect to the inner product $g_{\nu(p)}$ and $F^*(\nu(p)) = 1$. Thus, we call $\nu(p)$ an anisotropic unit normal vector of T_pM.

3. A connection on hypersurfaces of Minkowski space

Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ and denote $\nu : M \to W_F$ its anisotropic Gauss map.

Let $\overline{\nabla}$ be the standard connection on the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. For vector fields X, Y on M, we decompose $\overline{\nabla}_X Y$ as the tangent part $\nabla_X Y$ and the anisotropic normal part $\mathrm{II}(X, Y)\nu$ with respect to the inner product g_ν. That is:

$$\overline{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y)\nu, \tag{3.1}$$

where $g_\nu(\nabla_X Y, \nu) = 0$.

We also have the Weingarten formula:

$$\overline{\nabla}_X \nu = -S_F X, \tag{3.2}$$

and

$$g_\nu(S_F X, Y) = \mathrm{II}(X, Y), \tag{3.3}$$

where we have used (2.4).

It is easy to verify that ∇ is a torsion free connection on M and II is a symmetric second order covariant tensor field on M. We call II the anisotropic second fundamental form.

Let $\{e_i\}_{i=1}^n$ be a local frame of M and $\{\omega^i\}_{i=1}^n$ its dual frame. Let $g_{ij} = g_\nu(e_i, e_j)$, $\nabla e_i = \omega_i^j \otimes e_j$, $\mathrm{II}(e_i, e_j) = h_{ij}$, $h_i^j = g^{jk} h_{ki}$, where (g^{ij}) is the inverse matrix of (g_{ij}). Then we have

$$dx = \omega^i e_i, \tag{3.4}$$

$$de_i = \omega_i^j e_j + h_{ij}\omega^j \nu, \tag{3.5}$$

$$d\nu = -h_i^j \omega^i e_j. \tag{3.6}$$

Differentiate (3.4) and using (3.5), we get

$$d\omega^i = \omega^j \wedge \omega_j^i, \tag{3.7}$$

$$h_{ij} = h_{ji}. \tag{3.8}$$

Differentiate (3.5) and using (3.5-3.6), we get

$$h_{ijk} = h_{ikj}, \tag{3.9}$$

$$d\omega_i^j - \omega_i^k \wedge \omega_k^j = -\frac{1}{2} R_i{}^j{}_{kl} \omega^k \wedge \omega^l, \tag{3.10}$$

where

$$dh_{ij} - h_{ik}\omega_j^k - h_{kj}\omega_i^k = h_{ijk}\omega^k,$$

and $R_i{}^j{}_{kl} = -R_i{}^j{}_{lk} = h_{ik}h_l^j - h_{il}h_k^j$.

Differentiate (3.6) and using (3.5), we get

$$h_i{}^j{}_k = h_k{}^j{}_i, \tag{3.11}$$

where

$$dh_i^j + h_i^k\omega_k^j - h_k^j\omega_i^k = h_i{}^j{}_k\omega^k.$$

Note (h_i^j) is the matrix of the F-Weingarten operator $S_F = -d\nu$, its eigenvalues are called the anisotropic principal curvatures, we denote them by $\kappa_1, \cdots, \kappa_n$.

We have n invariants, the elementary symmetric function σ_r of the anisotropic principal curvatures:

$$\sigma_r = \sum_{i_1 < \cdots i_r} \kappa_{i_1} \cdots \kappa_{i_n} \quad (1 \leq r \leq n). \tag{3.12}$$

For convenience, we set $\sigma_0 = 1$. The r-th anisotropic mean curvature $H_{r;F}$ is defined by

$$H_{r;F} = \sigma_r / C_n^r, \quad C_n^r = \frac{n!}{r!(n-r)!}. \tag{3.13}$$

Using the characteristic polynomial of S_F, σ_r is defined by

$$\det(tI - S_F) = \sum_{r=0}^{n} (-1)^r \sigma_r t^{n-r}. \tag{3.14}$$

So, we have

$$\sigma_r = \frac{1}{r!} \sum_{i_1,\cdots,i_r;j_1,\cdots,j_r} \delta_{i_1\cdots i_r}^{j_1\cdots j_r} h_{j_1}^{i_1} \cdots h_{j_r}^{i_r}, \tag{3.15}$$

where $\delta_{i_1\cdots i_r}^{j_1\cdots j_r}$ is the usual generalized Kronecker symbol, i.e., $\delta_{i_1\cdots i_r}^{j_1\cdots j_r}$ equals +1 (resp. -1) if $i_1 \cdots i_r$ are distinct and $(j_1 \cdots j_r)$ is an even (resp. odd) permutation of $(i_1 \cdots i_r)$ and in other cases it equals zero.

Definition 3.1. Let $f : M \to \mathbb{R}$ be a smooth function. We define the gradient (with respect to the induced metric g_ν on M) $\operatorname{grad} f$ of the function f by

$$g_\nu(\operatorname{grad} f, X) = X(f), \tag{3.16}$$

where X is any smooth vector field on M.

Define f_i by $df = f_i\omega^i$, then

$$\operatorname{grad} f = g^{ij} f_j e_i. \tag{3.17}$$

We define

$$dV = |e_1, \cdots, e_n, \nu|\, \omega^1 \wedge \cdots \wedge \omega^n,$$

where $|e_1, \cdots, e_n, \nu|$ is the determinant of the matrix $(e_1, \cdots, e_n, \nu)$. Then dV is a volume element on M.

Definition 3.2. Let X be a smooth vector field on M. We define the divergence (with respect to the volume element dV) $\operatorname{div} X$ by $d\{i(X)dV\} = (\operatorname{div} X)dV$, where

$$(i(X)dV)(Y_1, \cdots, Y_{n-1}) \equiv dV(X, Y_1, \cdots, Y_{n-1}), \qquad \forall Y_1, \cdots, Y_{n-1} \in \mathscr{X}(M).$$

Lemma 3.3. *Let $X = X^i e_i$, then $\operatorname{div} X = X^i_i$, where*

$$dX^i + X^j \omega^i_j = X^i_j \omega^j.$$

Proof. By (3.5), (3.6), we get

$$d\, |e_1, \cdots, e_n, \nu| = \omega^i_i\, |e_1, \cdots, e_n, \nu|\, . \tag{3.18}$$

From the definition of $i(X)$, we have

$$i(X)dV = \sum_i (-1)^{i+1} X^i\, |e_1, \cdots, e_n, \nu|\, \omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^n.$$

So,

$$\begin{aligned} & d\{i(X)dV\} = \textstyle\sum_i (-1)^{i+1} (dX^i) \wedge |e_1, \cdots, e_n, \nu|\, \omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^n \\ + \; & \textstyle\sum_i (-1)^{i+1} X^i (d\, |e_1, \cdots, e_n, \nu|) \wedge \omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^n \\ + \; & \textstyle\sum_{j<i} (-1)^{i+j} X^i\, |e_1, \cdots, e_n, \nu|\, d\omega^j \wedge \omega^1 \wedge \cdots \wedge \widehat{\omega^j} \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^n \\ + \; & \textstyle\sum_{j>i} (-1)^{i+j+1} X^i\, |e_1, \cdots, e_n, \nu|\, d\omega^j \wedge \omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \widehat{\omega^j} \wedge \cdots \wedge \omega^n \\ = \; & X^i_i dV. \end{aligned}$$

□

4. $L_{r;F}$ OPERATOR FOR HYPERSURFACES

We introduce the Newton transformation defined by

$$P_r = \sigma_r I - \sigma_{r-1} S_F + \cdots + (-1)^r S_F^r, \quad r = 0, \cdots, n,$$

then

$$P_0 = I,\ P_n = 0,\ P_r = \sigma_r I - P_{r-1} S_F.$$

Lemma 4.1. *The matrix of P_r is given by:*

$$(P_r)_i^j = \frac{1}{r!}\delta_{i_1\cdots i_r i}^{j_1\cdots j_r j} h_{j_1}^{i_1}\cdots h_{j_r}^{i_r}. \tag{4.1}$$

Proof. We prove Lemma 4.1 inductively. For $r=0$, it is easy to check that (4.1) is true.

We can check directly

$$\delta_{i_1\cdots i_q}^{j_1\cdots j_q} = \begin{vmatrix} \delta_{i_1}^{j_1} & \delta_{i_1}^{j_2} & \cdots & \delta_{i_1}^{j_{q-1}} & \delta_{i_1}^{j_q} \\ \delta_{i_2}^{j_1} & \delta_{i_2}^{j_2} & \cdots & \delta_{i_2}^{j_{q-1}} & \delta_{i_2}^{j_q} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{i_{q-1}}^{j_1} & \delta_{i_{q-1}}^{j_2} & \cdots & \delta_{i_{q-1}}^{j_{q-1}} & \delta_{i_{q-1}}^{j_q} \\ \delta_{i_q}^{j_1} & \delta_{i_q}^{j_2} & \cdots & \delta_{i_q}^{j_{q-1}} & \delta_{i_q}^{j_q} \end{vmatrix} \tag{4.2}$$

Assume that (4.1) is true for $r=k$, we only need to show that it is also true for $r=k+1$. For $r=k+1$, Using (3.15) and (4.2), we have

$$\begin{aligned} RHS \text{ of } (4.1) &= \frac{1}{(k+1)!}\sum_{i_1,\cdots,i_{k+1};j_1,\cdots,j_{k+1}} \delta_{i_1\cdots i_{k+1} i}^{j_1\cdots j_{k+1} j} h_{i_1}^{j_1}\cdots h_{i_{k+1}}^{j_{k+1}} \\ &= \frac{1}{(k+1)!}\sum \begin{vmatrix} \delta_{i_1}^{j_1} & \delta_{i_1}^{j_2} & \cdots & \delta_{i_1}^{j_{k+1}} & \delta_{i_1}^{j} \\ \delta_{i_2}^{j_1} & \delta_{i_2}^{j_2} & \cdots & \delta_{i_2}^{j_{k+1}} & \delta_{i_2}^{j} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{i_{k+1}}^{j_1} & \delta_{i_{k+1}}^{j_2} & \cdots & \delta_{i_{k+1}}^{j_{k+1}} & \delta_{i_{k+1}}^{j} \\ \delta_{i}^{j_1} & \delta_{i}^{j_2} & \cdots & \delta_{i}^{j_{k+1}} & \delta_{i}^{j} \end{vmatrix} h_{i_1}^{j_1}\cdots h_{i_{k+1}}^{j_{k+1}} \\ &= \frac{1}{(k+1)!}\sum(\delta_i^j\delta_{i_1\cdots i_{k+1}}^{j_1\cdots j_{k+1}} - \delta_i^{j_{k+1}}\delta_{i_1\cdots i_k i_{k+1}}^{j_1\cdots j_k j} + \cdots)h_{i_1}^{j_1}\cdots h_{i_{k+1}}^{j_{k+1}} \\ &= \sigma_{k+1}\delta_i^j - \frac{1}{(k+1)!}\sum \delta_i^{j_{k+1}}\delta_{i_1\cdots i_k i_{k+1}}^{j_1\cdots j_k j} h_{i_1}^{j_1}\cdots h_{i_{k+1}}^{j_{k+1}} + \cdots \\ &= \sigma_{k+1}\delta_i^j - \sum (P_k)_i^{i_{k+1}} h_{i_{k+1}}^j \\ &= (P_{k+1})_i^j. \end{aligned}$$

□

Lemma 4.2. *For each r, we have*

(a) $(P_r)_i{}^j{}_j = 0$*;*

(c) $\mathrm{Trace}(P_r S_F) = (r+1)\sigma_{r+1}$*;*

(c) $\mathrm{Trace}(P_r) = (n-r)\sigma_r$*;*

(d) $\mathrm{Trace}(P_r S_F^2) = \sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2}$.

Proof. (a). Noting (j, j_r) is skew symmetric in $\delta_{i_1\cdots i_r i}^{j_1\cdots j_r j}$ and (j, j_r) is symmetric in $h_{j_1}^{i_1}\cdots h_{j_r}^{i_r}{}_j$ (from (3.11)), we have

$$\sum_j (P_r)_i{}^j{}_j = \frac{1}{(r-1)!} \sum_{i_1,\cdots,i_r;j_1,\cdots,j_r;j} \delta^{j_1\cdots j_r j}_{i_1\cdots i_r i} h^{i_1}_{j_1} \cdots h_{j_r}{}^{i_r}{}_j = 0.$$

(b). Using (4.1) and (3.15), we have

$$\begin{aligned} \mathrm{Trace}(P_r S_F) &= \textstyle\sum_{ij} (P_r)^j_i h^i_j \\ &= \tfrac{1}{r!} \textstyle\sum_{i_1,\cdots,i_r;j_1,\cdots,j_r;i,j} \delta^{j_1\cdots j_r j}_{i_1\cdots i_r i} h^{i_1}_{j_1} \cdots h^{i_r}_{j_r} h^i_j \\ &= (r+1)\sigma_{r+1}. \end{aligned}$$

(c). Using (b) and the definition of P_r, we have

$$\mathrm{Trace}(P_r) = \mathrm{tr}(\sigma_r I) - \mathrm{tr}(P_{r-1} S_F) = n\sigma_r - r\sigma_r = (n-r)\sigma_r.$$

(d). Using (b) and the definition of P_{r+1}, we have

$$\mathrm{Trace}(P_r S_F^2) = \mathrm{Trace}(\sigma_{r+1} S_F) - \mathrm{Trace}(P_{r+1} S_F) = \sigma_1 \sigma_{r+1} - (r+2)\sigma_{r+2}.$$

□

Remark 4.3. When $F = 1$, Lemma 4.2 was a well-known result (for example, see Barbosa-Colares [5], or Reilly [26]).

Lemma 4.4.

$$(\sigma_r)_k = \sum_{i,j} (P_{r-1})^j_i h_j{}^i{}_k. \tag{4.3}$$

Proof. From the definition of σ_r, we have the following calculation:

$$\begin{aligned} (\sigma_r)_k &= \tfrac{1}{r!} \textstyle\sum_{i_1,\cdots,i_r;j_1,\cdots,j_r} \delta^{j_1\cdots j_r}_{i_1\cdots i_r} (h^{j_1}_{i_1} \cdots h^{j_r}_{i_r})_k \\ &= \tfrac{1}{(r-1)!} \textstyle\sum_{i_1,\cdots,i_r;j_1,\cdots,j_r} \delta^{j_1\cdots j_r}_{i_1\cdots i_r} h^{j_1}_{i_1} \cdots h_{i_r}{}^{j_r}{}_k \\ &= \sum_{i_r,j_r} (P_{r-1})^{j_r}_{i_r} h_{i_r}{}^{j_r}{}_k = \sum_{i,j} (P_{r-1})^j_i h_i{}^j{}_k. \end{aligned}$$

□

We define an operator $L_{r;F} : C^\infty(M) \to C^\infty(M)$ by

$$L_{r;F}(f) = \mathrm{div}(P_r \nabla f). \tag{4.4}$$

In the sequel, we will need the following lemma. Item (a) is essentially the content of lemma 1.1 and equation (1.3) in [12], while item (b) is quoted as proposition 1.5 in [13].

Lemma 4.5. *Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface, and $0 \le r \le n-1$, $p \in M$.*

(a) If $\sigma_{r+1}(p) = 0$, then P_r is semi-definite at p;

(b) If $\sigma_{r+1}(p) = 0$ and $\sigma_{r+2}(p) \neq 0$, then P_r is definite at p.

Another important result is (see [7]):

Lemma 4.6. *Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface, and $p \in M$.*

(a) For $1 \le r \le n$, one has $H_{r;F}^2 \ge H_{r-1;F}H_{r+1;F}$. Moreover, if equality happens for $r = 1$ or for some $1 < r < n$, with $H_{r+1;F} \neq 0$ in this case, then p is anisotropic umbilical point (i.e. $\kappa_1(p) = \kappa_2(p) = \cdots = \kappa_n(p)$);

(b) If, for some $1 \le r < n$, one has $H_{r;F} = H_{r+1;F} = 0$, then $H_{j;F} = 0$ for all $r \le j \le n$. In particular, at most $r-1$ of the anisotropic principal curvatures are different from zero.

The result below is standard, so we omit the proof.

Lemma 4.7. *Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface. The operator $L_{r;F}$ associated to the immersion x is elliptic if and only if P_r is positive definitive.*

Definition 4.8. Let $f : M \to \mathbb{R}$ be a smooth function. The Laplacian Δf is defined by $\Delta f := L_{0;F} f = \mathrm{div}(\mathrm{grad}\, f)$.

It is easy to see Δ is a elliptic differential operator.

Definition 4.9. Let $x : M \to \mathbb{R}^{n+1}$ be an immersed hypersurface, ν its anisotropic unit normal vector field. The function $u := g_\nu(x, \nu)$ is called the support function of the immersion x.

Next, we compute $L_{r;F}u$ for the support function $u = g_\nu(x, \nu)$.

Differentiate the decomposition

$$x = g^{ij} g_\nu(x, e_i)e_j + u\nu, \tag{4.5}$$

we obtain

$$dx = \{d(g^{ij} g_\nu(x, e_j))\}e_i + g^{ij} g_\nu(x, e_j)de_i + (du)\nu + ud\nu. \tag{4.6}$$

So, from (3.4), (3.5) and (3.6) we have

$$\omega^i e_i = \{d(g^{ij} g_\nu(x, e_j)) + g^{kj} g_\nu(x, e_j)\omega_k^i - uh_j^i\omega^j\}e_i + (du + g^{jk} g_\nu(x, e_j)h_{ik}\omega^i)\nu.$$

Thus, we get

$$du = -g^{jk} g_\nu(x, e_j)h_{ik}\omega^i,$$

and

$$d(g^{ij}g_\nu(x,e_j)) + g^{kj}g_\nu(x,e_j)\omega_k^i - uh_j^i\omega^j = \omega^i.$$

Denote u^i, $(g^{ij}g_\nu(x,e_j))_k$, u_j^i by

$$\operatorname{grad} u = u^i e_i,$$

$$(g^{ij}g_\nu(x,e_j))_k\omega^k = d(g^{ij}g_\nu(x,e_j)) + (g^{kj}g_\nu(x,e_j))\omega_k^i,$$

$$u_j^i\omega^j = du^i + u^j\omega_j^i$$

respectively. Then we have (using (3.11)):

$$u^i = -g^{il}h_{kl}g^{jk}g_\nu(x,e_j) = -h_k^i g^{kl}g_\nu(x,e_l),$$

$$(g^{ik}g_\nu(x,e_k))_j = \delta_j^i + h_j^i g_\nu(x,\nu),$$

$$u_j^i = -h_k{}^i{}_j g^{kl}g_\nu(x,e_l) - h_k^i(g^{kl}g_\nu(x,e_l))_j = -h_j{}^i{}_k g^{kl}g_\nu(x,e_l) - h_j^i - h_k^i h_j^k u.$$

By using Lemma 4.2 and Lemma 4.4, we get

$$\begin{aligned} L_{r;F}u &= (P_r)_i^j u_j^i = -(P_r)_i^j h_j{}^i{}_k g^{kl}g_\nu(x,e_l) - (P_r)_i^j h_j^i - (P_r)_i^j h_k^i h_j^k u \\ &= -(\sigma_{r+1})_k g^{kl}g_\nu(x,e_l) - (P_r)_i^j h_j^i - (P_r)_i^j h_k^i h_j^k u \\ &= -g_\nu(\nabla\sigma_{r+1},x) - (r+1)\sigma_{r+1} - (\sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2})u. \end{aligned}$$

Thus, we proved the following lemma:

Lemma 4.10. *For $0 \le r \le n-1$, we have:*

$$L_{r;F}u = -g_\nu(\nabla\sigma_{r+1},x) - (r+1)\sigma_{r+1} - (\sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2})u. \tag{4.7}$$

Remark 4.11. Recall $\sigma_1 = nH_F$ and $|\mathrm{II}|^2 = \sigma_1^2 - 2\sigma_2$, let $r=0$ in (4.7) we get

$$\Delta u = -n(H_F + g_\nu(\operatorname{grad} H_F, x)) - |\mathrm{II}|^2 u. \tag{4.8}$$

5. Proof of Theorem 1.4 and Theorem 1.5

We fix a point $o \in W$ as the origin of $\mathbb{R}^{n+1}$. Without loss of generality, we assume, for each $p \in M$, $\nu(p)$ be the anisotropic unit normal vector of $x(M)$ at $x(p)$ such that $\langle x(p),\nu(p)\rangle_{\nu(p)} > 0$ (otherwise we consider the function $-u$ instead). This gives an orientation to M, indeed, the component of the position vector x perpendicular(with respect to the inner product g_ν) to M defines a never zero, anisotropic normal, vector field on M, such that the support function $u = \langle x(p),\nu(p)\rangle_{\nu(p)}$ is positive on M.

5.1. **Proof of Theorem 1.4.** Since x is anisotropic minimal, from (4.8) we get

$$\Delta u = -|\mathrm{II}|^2 u \leq 0, \text{ on } M. \tag{5.1}$$

Let $u_* = \inf_M u$. We claim that u_* is attained at some point $x_0 \in M$. Consider a sequence $\{x_k\} \subset M$ such that $u(x_k) \to u_*$ as $k \to +\infty$. To each x_k we associate $y_k = u(x_k)\nu(x_k)$, then $y_k \in T_{x_k}M$. Since $\|y_k\|_{\mathbb{R}^{n+1}} = u(x_k)\|\nu(x_k)\|_{\mathbb{R}^{n+1}}$ is bounded, there exists a subsequence, which again we call $\{y_k\}$, such that $y_k \to y_0$ for some $y_0 \in \mathbb{R}^{n+1}$. Since $\bigcup_{p\in M} T_pM$ is closed and $\{y_k\} \subset_{p\in M} T_pM$ we deduce $y_0 \in T_{x_0}M$ for some $x_0 \in M$. Thus, by the continuity of F^* and Lemma 2.2,

$$u_* = \lim_{k\to+\infty} u(x_k) = \lim_{k\to+\infty} F^*(y_k) = F^*(y_0) \geq g_{\nu(x_0)}(y_0, \nu(x_0)) = u(x_0),$$

so $u^* = u(x_0)$ as needed. Now, from the usual maximum principle u is constant, $u = u_* = u(x_0) > 0$. From (5.1) we then have $\mathrm{II} \equiv 0$ and x is totally geodesic.

5.2. **Proof of Theorem 1.5.** Since $H_{r+1;F} = 0$, from Lemma 4.10 we get

$$L_{r;F}u = (r+2)\sigma_{r+2}u. \tag{5.2}$$

Using Lemma 4.5(a) we have that P_r is semi-definite. Since $H_{r;F}$ does not vanish, we have that $H_{r;F}$ is positive or negative, because $c(r)H_{r;F} = \mathrm{Trace}(P_r)$, where $c(r) = (n-r)C_n^r$. Now we use Lemma 4.6 and obtain:

$$0 = H_{r+1;F}^2 \geq H_{r;F}H_{r+2;F}. \tag{5.3}$$

Using the information above, we claim that $H_{r+2;F} \equiv 0$.

Case(i) $H_{r;F} > 0$.

In this case, P_r is positive defined, and $L_{r;F}$ is elliptic by Lemma 4.7. Using (5.3) we conclude that $H_{r+2;F} \leq 0$. Whereas from (5.2) we have

$$L_{r;F}u \leq 0.$$

Following exactly the proof as in Theorem 1.4, we conclude that u is constant, $u = u_* = u(x_0) > 0$. From (5.2) we then have $H_{r+2;F} \equiv 0$.

Case(ii) $H_{r;F} < 0$.

In this case, P_r is negative defined, and $-L_{r;F}$ is elliptic by Lemma 4.7. Using (5.3) we conclude that $H_{r+2;F} \geq 0$. Whereas from (5.2) we have

$$-L_{r;F}u \leq 0.$$

Now, following exactly the proof as in Theorem 1.4, we conclude that u is constant, $u = u_* = u(x_0) > 0$. From (5.2) we then have $H_{r+2;F} \equiv 0$.

Thus we conclude that $H_{r+2;F} \equiv 0$. Now, we use Lemma 4.6(b) to conclude that $H_{j;F} = 0$ for $j \geq r+1$ and so that $v \geq n-r$. Since $H_{r;F}$ does not change sign we have that $v = n-r$.

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